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REAL AND GAUSSIAN INTEGER SOLUTIONS TO $x^2 + y^2 = 2(z^2 - w^2)$

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ABSTRACT

The quadratic equation with four unknowns given by $x^2 + y^2 = 2(z^2 - w^2)$ is analysed for its non-zero distinct integer solutions and Gaussian integer solutions. Different choices of solutions in real and Gaussian integers are obtained. A general formula for obtaining sequence of solutions (real and complex) based on its given solution is illustrated.

Keywords*: Quadratic with four unknowns, real integers, Gaussian integers*

I. INTRODUCTION

Number theory is the branch of Mathematics concerned with studying the properties and relations of integers. There are number of branches of number theory of which Diophantine equation is very important. Diophantine equations are numerically rich because of their variety [1-3]. In [4-11], different patterns of integer solutions to quadratic Diophantine equation with four unknowns are discussed. In [12], Gaussian integer solutions to space Pythagorean equation are obtained. In this communication, the quadratic equation with four unknowns given by $x^2 + y^2 = 2(z^2 - w^2)$ is analysed for its non-zero distinct integer solutions and Gaussian integer solutions.

II. METHOD OF ANALYSIS

2.1 Section: A (Real integer solutions)

The quadratic equation with four unknowns to be solved is

$$x^2 + y^2 = 2(z^2 - w^2) \quad (1)$$

Introduction of the linear transformations

$$x = u + v, \quad y = u - v \quad (2)$$

in (1) leads to

$$u^2 + v^2 + w^2 = z^2 \quad (3)$$

which is in the form of space Pythagorean equation

The choices of solutions for (3) are represented below:

i) $u = m^2 - n^2 - p^2 + q^2$, $v = 2mn - 2pq$,
$w = 2mp + 2nq$, $z = m^2 + n^2 + p^2 + q^2$

ii) $u = 2mp + 2nq$, $v = 2mn - 2pq$,
$w = m^2 - n^2 - p^2 + q^2$,
$z = m^2 + n^2 + p^2 + q^2$

iii) $u = 2mp + 2nq$, $v = m^2 - n^2 - p^2 + q^2$,
$w = 2mn - 2pq$, $z = m^2 + n^2 + p^2 + q^2$

iv) $u = 2ab$, $v = 2ac$, $w = a^2 - b^2 - c^2$,
$z = a^2 + b^2 + c^2$

v) $u = a^2 - b^2 - c^2$, $v = 2ac$, $w = 2ab$,
$z = a^2 + b^2 + c^2$

In view of (2), one may obtain different sets of solutions to (1) which are presented below:

Set: 1

Considering choice (i), the general solution of (1) is

$$x = m^2 - n^2 - p^2 + q^2 + 2mn - 2pq$$
$$y = m^2 - n^2 - p^2 + q^2 - 2mn + 2pq$$
$$z = m^2 + n^2 + p^2 + q^2$$
$$w = 2mp + 2nq$$

Set: 2

For choice (ii), the general solution of (1) is

$$x = 2m(p+n) + 2q(n-p)$$
$$y = 2m(p-n) + 2q(n+p)$$
$$z = m^2 + n^2 + p^2 + q^2$$
$$w = m^2 - n^2 - p^2 + q^2$$

Set: 3

For choice (iii), the general solution of (1) is

$$x = m^2 - n^2 - p^2 + q^2 + 2mp + 2nq$$
$$y = -m^2 + n^2 + p^2 - q^2 + 2mp + 2nq$$
$$z = m^2 + n^2 + p^2 + q^2$$
$$w = 2mn - 2pq$$

Set: 4

For choice (iv), the general solution of (1) is

$$x = 2a(b+c)$$
$$y = 2a(b-c)$$
$$z = a^2 + b^2 + c^2$$
$$w = a^2 - b^2 - c^2$$

Set: 5

For choice (v), the general solution of (1) is

$$x = a^2 - b^2 - c^2 + 2ac$$
$$y = a^2 - b^2 - c^2 - 2ac$$
$$z = a^2 + b^2 + c^2$$
$$w = 2ab$$

In addition to the above sets of solutions to (1), there are other representations of solutions to (1) which are illustrated below:

2.2 Representation: 1

Write (3) as

$$u^2+v^2=z^2-w^2 \quad (4)$$

Assume $\alpha^2=z^2-w^2$ (5)

Rewrite (5) as

$$z^2=\alpha^2+w^2 \quad (6)$$

which is in the form of Pythagorean equation satisfied by the following two sets of solutions

Set: 1 $z=r^2+s^2$, $w=2rs$, $\alpha=r^2-s^2$, $r>s>0$ (7)

Set: 2 $z=r^2+s^2$, $w=r^2-s^2$, $\alpha=2rs$, $r>s>0$ (8)

Consider (7). Using (7) in (4), we have

$$u^2+v^2=(r^2-s^2)^2 \quad (9)$$

which is satisfied by

$$r=f^2+g^2+h^2$$
$$s=f^2-g^2-h^2$$
$$u=8f^2gh$$
$$v=4f^2(g^2-h^2)$$

In view of (2) and (7), the corresponding non-zero distinct integral solutions of (1) are given by

$$x=4f^2(2gh+g^2-h^2)$$
$$y=4f^2(2gh-g^2+h^2)$$
$$z=2(f^4+g^4+h^4+2g^2h^2)$$
$$w=2(f^4-g^4-h^4-2g^2h^2)$$

Consider (8). Using (8) in (4), we have

$$u^2+v^2=(2rs)^2 \quad (10)$$

which is in the form of Pythagorean equation satisfied by

$$u=p^2-q^2 \quad (11)$$
$$v=2pq \quad (12)$$
$$2rs=p^2+q^2 \quad (13)$$

Here, the equation (13) is satisfied for the following choices of r and s:

i) $s=1, r=k^2+2k+2$

ii) $s=2, r=2k^2+2k+1$

iii) $s=2, r=2k^2+4k+4$

Considering choice (i) and performing simplification, the corresponding solutions to (1) are given by

$$x=2k^2+8k+4$$
$$y=4-2k^2$$
$$z=k^4+4k^3+8k^2+8k+5$$
$$w=k^4+4k^3+8k^2+8k+3$$

Similarly for choice (ii), the general solutions to (1) are given by

$$x=8k^2+16k+4$$
$$y=4-8k^2$$
$$z=4k^4+8k^3+8k^2+4k+5$$

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$$w = 4k^4 + 8k^3 + 8k^2 + 4k - 3$$

For choice (iii), the general solutions to (1) are found to be

$$x = 8k^2 + 32k + 16$$
$$y = 16 - 8k^2$$
$$z = 4k^4 + 16k^3 + 32k^2 + 32k + 20$$
$$w = 4k^4 + 16k^3 + 32k^2 + 32k + 12$$

2.3 Representation: 2

The assumption $\alpha^3 = z^2 - w^2$ (14)
is equivalent to the following system of double equations:

System	$z+w$	$z-w$
1	α^2	α
2	α^3	1

Considering System: 1, it is seen that there are two sets of solutions to (1) represented respectively below:

Set: 1

$$x = (m+n)(m^2+n^2)$$
$$y = (m-n)(m^2+n^2)$$
$$z = \frac{1}{2}(m^2+n^2)(m^2+n^2+1)$$
$$w = \frac{1}{2}(m^2+n^2)(m^2+n^2-1)$$

Set: 2

$$x = m^2(m+3n) - n^2(3m+n)$$
$$y = m^2(m-3n) - n^2(3m-n)$$
$$z = \frac{1}{2}(m^2+n^2)(m^2+n^2+1)$$
$$w = \frac{1}{2}(m^2+n^2)(m^2+n^2-1)$$

Similarly, Considering System: 2, it is seen that there are two sets of solutions to (1) represented respectively below:

Set: 3

$$x = (m+n)(m^2+n^2)$$
$$y = (m-n)(m^2+n^2)$$
$$z = \frac{1}{2}\left[(m^2+n^2)^3+1\right]$$
$$w = \frac{1}{2}\left[(m^2+n^2)^3-1\right]$$

Set: 4

$$x = m^2(m+3n) - n^2(3m+n)$$
$$y = m^2(m-3n) - n^2(3m-n)$$
$$z = \frac{1}{2}\left[(m^2+n^2)^3+1\right]$$

$$w=\frac{1}{2}\left[\left(m^2+n^2\right)^3-1\right]$$

It is worth to note that m and n should be of different parity. Otherwise, the values of z and w are not in integers.

2.4 Representation: 3

Substituting $z=\frac{\alpha(\alpha+1)}{2}$ and $w=\frac{\alpha(\alpha-1)}{2}$ (15)

in (1), we get

$$x^2+y^2=2\alpha^3 \quad (16)$$

Assume $\alpha=p^2+q^2$, $p,q>0$ (17)

Write 2 as

$$2=(1+i)(1-i) \quad (18)$$

Substituting (17), (18) in (16) and employing the method of factorization, define

$$x+iy=(1+i)(p+iq)^3$$

Equating real and imaginary parts, we have

$$\left.\begin{aligned} x&=p^3-3p^2q-3pq^2+q^3 \\ y&=p^3+3p^2q-3pq^2-q^3 \end{aligned}\right\} \quad (19)$$

In view of (15), we have

$$\left.\begin{aligned} z&=\frac{1}{2}\left(p^2+q^2\right)\left(p^2+q^2+1\right) \\ w&=\frac{1}{2}\left(p^2+q^2\right)\left(p^2+q^2-1\right) \end{aligned}\right\} \quad (20)$$

Thus (19) and (20) represents non-zero distinct integral solutions to (1).

Note:

Instead of (18), one may write 2 as $2=\frac{(7+i)(7-i)}{25}$, $2=\frac{(1+7i)(1-7i)}{25}$

Following the procedure similar to above, one may obtain different sets of integral solutions to (1).

2.5 Representation: 4

Substituting $z=\frac{\alpha^3+1}{2}$ and $w=\frac{\alpha^3-1}{2}$ (21)

in (1), we get (16).

Using (17) in (21)

$$\left.\begin{aligned} z&=\frac{1}{2}\left[\left(p^2+q^2\right)^3+1\right] \\ w&=\frac{1}{2}\left[\left(p^2+q^2\right)^3-1\right] \end{aligned}\right\} \quad (22)$$

Hence (19) and (22) represents non-zero distinct integral solutions to (1). It is worth to note that p and q should be of different parity. Otherwise, the values of z and w are not in integers.

III. SECTION: B

Gaussian integer solutions

The substitution

$x=a+i2b$, $y=2a-ic$, $z=b+ia$, $w=c+ia$ (23)

in (1) leads to

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$$5a^2+c^2=6b^2 \qquad (24)$$

(24) is solved through three different methods and thus we obtain three different sets of Gaussian integer solutions to (1)

3.1 Method: 1

Write (24) in the form of ratio as

$$\frac{5(a+b)}{b+c}=\frac{b-c}{a-b}=\frac{m}{n}, \quad n\neq 0 \qquad (25)$$

which is equivalent to the system of double equations

$$5na+(5n-m)b-mc=0$$

$$-ma+(m+n)b-nc=0$$

Applying the method of cross multiplication, we get

$$a=m^2-5n^2+2mn \qquad (26)$$

$$b=m^2+5n^2 \qquad (27)$$

$$c=5n^2-m^2+10mn \qquad (28)$$

In view of (23), the corresponding non-zero distinct Gaussian integer solutions of (1) are given by

$$x=m^2-5n^2+2mn+i(2m^2+10n^2) \qquad y=2m^2-10n^2+4mn-i(5n^2-m^2+10mn)$$

$$z=m^2+5n^2+i(m^2-5n^2+2mn) \qquad w=5n^2-m^2+10mn+i(m^2-5n^2+2mn)$$

3.2 Method: 2

Assume $b=5p^2+q^2$, $p,q>0$ (29)

Write 6 as

$$6=(\sqrt{5}+i)(\sqrt{5}-i) \qquad (30)$$

Substituting (29), (30) in (24) and employing the method of factorization, define

$$\sqrt{5}a+ic=(\sqrt{5}+i)(\sqrt{5}p+iq)^2 \qquad (31)$$

Equating real and imaginary parts, we get

$$\left.\begin{array}{l} a=5p^2-q^2-2pq \\ c=5p^2-q^2+10pq \end{array}\right\} \qquad (32)$$

Using (29), (32) in (23), the corresponding non-zero distinct Gaussian integral solutions to (1) are found to be

$$x=5p^2-q^2-2pq+i(10p^2+2q^2) \qquad y=10p^2-2q^2-4pq-i(5p^2-q^2+10pq)$$

$$z=5p^2+q^2+i(5p^2-q^2-2pq) \qquad w=5p^2-q^2+10pq+i(5p^2-q^2-2pq)$$

3.3 Method: 3

One may write (24) as

$$6b^2-c^2=5a^2 \qquad (33)$$

Assume $a=6p^2-q^2$, $p,q>0$ (34)

Write 5 as

$$5=(\sqrt{6}+1)(\sqrt{6}-1) \qquad (35)$$

Substituting (34), (35) in (33) and employing the method of factorization, define

$$\sqrt{6}b+c=(\sqrt{6}+1)(\sqrt{6}p+q)^2 \qquad (36)$$

Equating rational and irrational parts, we get

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$$\left.\begin{array}{l} b=6p^2+q^2+2pq \\ c=6p^2+q^2+12pq \end{array}\right\} \qquad (37)$$

In view of (23), the corresponding non-zero distinct Gaussian integral solutions to (1) are given by

$$x=6p^2-q^2+i\left(12p^2+2q^2+4pq\right)\ y=2\left(6p^2-q^2\right)-i\left(6p^2+q^2+12pq\right)$$

$$z=6p^2+q^2+2pq+i\left(6p^2-q^2\right)\ w=6p^2+q^2+12pq+i\left(6p^2-q^2\right)$$

IV. GENERATION OF SOLUTIONS

Let (x_0, y_0, z_0) be the given integer solution to (1). Let (x_1, y_1, z_1) be the second solution of (1) where

$x_1 = h - x_0$, $y_1 = h - y_0$, $z_1 = z_0 + h$, $w_1 = h - w_0$ (38)

in which h is any non-zero integer to be determined.

Substituting (38) in (1) and simplifying, we have

$$h = x_0 + y_0 + 2z_0 + 2w_0$$

Thus, the second solution is given in the matrix form as

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \\ w_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 2 & 2 \\ 1 & 0 & 2 & 2 \\ 1 & 1 & 3 & 2 \\ 1 & 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \\ z_0 \\ w_0 \end{pmatrix}$$

Repeating the above process, the general solution to (1) in the matrix form as

$$\begin{pmatrix} x_n \\ y_n \\ z_n \\ w_n \end{pmatrix} = \begin{pmatrix} \frac{\tilde{y}_n + 3(-1)^n}{4} & \frac{\tilde{y}_n - (-1)^n}{4} & \tilde{x}_n & \frac{\tilde{y}_n - (-1)^n}{2} \\ \frac{\tilde{y}_n - (-1)^n}{4} & \frac{\tilde{y}_n + 3(-1)^n}{4} & \tilde{x}_n & \frac{\tilde{y}_n - (-1)^n}{2} \\ \frac{\tilde{x}_n}{2} & \frac{\tilde{x}_n}{2} & \tilde{y}_n & \tilde{x}_n \\ \frac{\tilde{y}_n - (-1)^n}{4} & \frac{\tilde{y}_n - (-1)^n}{4} & \tilde{x}_n & \frac{\tilde{y}_n + (-1)^n}{2} \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \\ z_0 \\ w_0 \end{pmatrix}$$

$, n = 1,2,3,\ldots$

where $(\tilde{x}_n, \tilde{y}_n)$ is the general solution of $y^2 = 2x^2 + 1$

given by

$$\tilde{y}_n = \frac{1}{2}\left[\left(3+2\sqrt{2}\right)^{n+1} + \left(3-2\sqrt{2}\right)^{n+1}\right]\ \tilde{x}_n = \frac{1}{2\sqrt{2}}\left[\left(3+2\sqrt{2}\right)^{n+1} - \left(3-2\sqrt{2}\right)^{n+1}\right], \quad n = 0, 1, 2, \ldots.$$

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