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SEMI-STRONG COLOR PARTITION OF A GRAPH

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ABSTRACT
Claude Berge introduced the concept of strong stable sets in a graph. A subset $S$ of a graph $G = (V, E)$ is a strong stable set if $|N[v] \cap S| \leq 1$ for every $v \in V(G)$. Relaxing this condition, Prof. E. Sampath kumar introduced semi-strong sets in graphs as those sets for which $|N(v) \cap S| \leq 1$ for every $v \in V(G)$. Resolvability is a well-studied concept. Combining these two, resolving semi-strong color partition is defined and studied in this paper.

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I. INTRODUCTION
A subset $S$ of a graph $G = (V, E)$ is called a semi-strong set if $|N[v] \cap S| \leq 1$ for every $v \in V(G)$.

A subset $S = \{x_1, x_2, x_3, \ldots, x_k\}$ of a connected graph $G$ is called a resolving set if the code $C(u : S) = (d(u, x_1), d(u, x_2), \ldots, d(u, x_k))$ is distinct for different $u$. A partition of $V(G)$ into subsets where each subset considered is a resolving semi-strong set. The minimum cardinality of such a partition denoted by $\chi_{spd}(G)$ is found out for some well-known graphs. Further, graphs with $\chi_{s}(G) = 2, \chi_{s}(G) = n$ are determined.

II. RESOLVING SEMI-STRONG COLOR PARTITION

Definition 1.1. Let $G$ be a finite, simple, connected, undirected graph. A partition $\Pi = \{V_1, V_2, \ldots, V_k\}$ is called a resolving semi-strong color partition if $\Pi$ is a semi-strong color partition and the k-vector $(v|\Pi) = (d(v, x_1), d(v, x_2), \ldots, d(v, x_k))$ is distinct for different $v$ in $V(G)$. The minimum cardinality of a resolving semi-strong color partition of $G$ is called semi-strong color class partition dimension of $G$ and is denoted by $\chi_{spd}(G)$. The trivial partition namely $\{\{v_1\}, \{v_2\}, \ldots, \{v_k\}\}$ where $V(G) = \{v_1, v_2, v_k\}$ is a resolving semi-strong color class partition of $G$.

Remark 1.2. (i) $\chi_{s}(G) \leq \chi_{spd}(G)$
(ii) $pd(G) \leq \chi_{spd}(G)$

Example 1.3. Let $G$ be the graph given in Fig.1.1: $\chi_{s}(G) = 5$. Therefore $\chi_{spd}(G) = 5$.

Figure 1.1

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Example 1.4. Let $G = P_n$. Let $V(P_n) = \{u_1, u_2, \ldots, u_n\}$, $n \geq 3$. Let $\Pi = \{\{u_1\}, \{u_2, u_3, u_6, u_7, \ldots\}, \{u_4\}, \{u_5\}, \{u_8\}, \ldots\}$. $\Pi$ is a minimum resolving semi-strong color partition of $P_n$. Therefore $\chi_{spd}(P_n) = 3$, $n \geq 4$. when $n = 1, 2, 3$ then, $\chi_{spd}(P_1) = 1$, $\chi_{spd}(P_2) = 2$, $\chi_{spd}(P_3) = 2$.

$\chi_{spd}(G)$ for some well-known Graphs

Proposition:

1. $\chi_{spd}(K_n) = n$.
2. $\chi_{spd}(K_1, n) = n$.
3. $\chi_{spd}(K_m, n) = \begin{cases} n & \text{if } m < n \\ n + 1 & \text{if } m = n \end{cases}$
4. $\chi_{spd}(W_n) = n$.
5. $\chi_{spd}(P) = 5$ where $P$ is the Petersen graph.

Proof: Let $V(P) = \{v_1, v_2, v_3, v_4, v_5, w_1, w_2, w_3, w_4, w_5\}$. Consider the Petersen graph given in Figure 1.2. Here $\chi_{spd}(P) = 5$. Also $\chi_{spd}(G) = 2$ if and only if $G = P_2$ or $P_3$.

$\chi_{spd}(C_4) = 3$ for every $n \geq 3$

Proof: Case 1: Let $n \equiv 0(\text{mod } 4)$. Let $n = 4k$.
Let $V(C_n) = \{u_1, u_2, \ldots, u_4k\}$. Let $\Pi = \{\{u_{4k-2}, u_{4k-1}\}, \{u_1, u_2, u_3, u_4, u_5, \ldots, u_{4k-7}, u_{4k-6}, u_{4k-3}\}, \{u_3, u_4, u_7, u_8, \ldots, u_{4k-5}, u_{4k-4}, u_{4k}\}\}$. Then $\Pi$ is a resolving semi-strong color partition of $C_n$. Therefore $\chi_{spd}(C_n) \leq 3$. Suppose $\chi_{spd}(C_n) = 1$. Then $n = 1$, a contradiction. If $\chi_{spd}(C_n) \neq 2$, since $\chi_{spd}(G) = 2$ if and only if $G = P_2$ or $P_3$. Therefore $\chi_{spd}(C_n) = 3$ when $n \equiv 0(\text{mod } 4)$.

Case 2: Let $n \equiv 1(\text{mod } 4)$. Let $n = 4k + 1$.
Let $V(C_n) = \{u_1, u_2, \ldots, u_{4k+1}\}$. Let $\Pi = \{\{u_{4k+2}, u_{4k+3}\}, \{u_1, u_2, u_3, u_4, \ldots, u_{4k-3}, u_{4k-2}\}, \{u_5, u_6, u_7, \ldots, u_{4k-1}, u_{4k}\}\}$. Then it can be easily verified that $\Pi$ is a resolving semi-strong color partition of $C_n$. Therefore, $\chi_{spd}(C_n) \leq 3$. But $\chi_{spd}(C_n) \geq 3$, since $\chi_{spd}(G) = 2$ if and only if $G = P_2$ or $P_3$ and $\chi_{spd}(G) = 1$ if and only if $G = K_1$. Therefore $\chi_{spd}(C_n) = 3$ when $n \equiv 1(\text{mod } 4)$.

Case 3: Let $n \equiv 2(\text{mod } 4)$. Let $n = 4k + 2$.
Let $V(C_n) = \{u_1, u_2, \ldots, u_{4k+2}\}$.

Figure 1.2
Let $\Pi = \{(u_{4k+1}, u_{4k+2}), \{u_1, u_2, u_3, u_6, \ldots, u_{4k-3}, u_{4k-2}\}, \{u_1, u_4, u_7, u_8, \ldots, u_{4k-1}, u_{4k}\}\}$. Then it can be easily verified that $\Pi$ is a resolving semi-strong color partition of $C_n$. Therefore, $\chi_{spd}(C_n) \leq 3$. But $\chi_{spd}(C_n) \geq 3$, since $\chi_{spd}(G) = 2$ if and only if $G = P_2$ or $P_3$ and $\chi_{spd}(G) = 1$ if and only if $G = K_1$.

Therefore $\chi_{spd}(C_n) = 3$ when $n \equiv 2(\mod 4)$.

**Case 4:** Let $n = 3(\mod 4)$. Let $n = 4k + 3$.

Let $V(C_n) = \{u_1, u_2, \ldots, u_{4k+3}\}$.

Let $\Pi = \{(u_{4k+1}, u_{4k+2}), \{u_1, u_2, u_3, u_6, \ldots, u_{4k-3}, u_{4k-2}, u_{4k+1}, u_{4k+2}\}, \{u_1, u_4, u_7, u_8, \ldots, u_{4k-1}, u_{4k}, u_{4k+3}\}\}$. Then it can be easily verified that $\Pi$ is a resolving semi-strong color partition of $C_n$. Therefore $\chi_{spd}(C_n) \leq 3$. But $\chi_{spd}(C_n) \geq 3$.

Therefore $\chi_{spd}(C_n) = 3$ when $n \equiv 3(\mod 4)$.

Hence $\chi_{spd}(C_n) = 3$ for every $n \geq 3$.

**Theorem 1.5.** $\chi_{spd}(G) = 2$ if and only if $G = P_2$ or $P_3$.

**Proof:** Let $V(G) = \{u_1, u_2, \ldots, u_k\}$. Let $\chi_{spd}(G) = 2$. Let $\Pi = \{V_1, V_2\}$ be a semi strong color class partition of $G$.

Then there exist vertices $u_i \in V_1, u_j \in V_2$ such that $u_i$ and $u_j$ are adjacent (since $G$ is connected). Suppose $u_i$ is adjacent with $v_1$ and $v_2$ in $V_2$, $r(v_1) = (1, 1)$, $r(v_2) = (0, 0)$, a contradiction. Therefore, $u_i$ is a unique vertex in $V_1$ that is adjacent to a vertex in $V_2$ and $u_j$ is the unique vertex in $V_2$ that is adjacent to a vertex in $V_1$. Suppose $|V_1| \geq 2$. Let $u_1 \in V_1$. Suppose $u_1$ is not adjacent with any vertex of $V_1$. Then $r(u_1) = (0, 1)$ and $r(u_1) = (1, 0)$, a contradiction. Therefore, $u_1$ is adjacent with at least one vertex of $V_1$ and not adjacent with any vertex of $V_2$. $u_i$ is adjacent with at most one vertex in $V_1$. For if $u_j$ is adjacent with $u_1, u_2 \in V_1$, then $r(u_1) = (0, 2)$ and $r(u_2) = (0, 2)$, a contradiction. Therefore, $u_i$ is adjacent with exactly one vertex in $V_1$, $w$. Let $w$ be a unique vertex in $V_1$ which is adjacent with $u_i$. If $w$ is adjacent with another vertex in $V_1$, then $V_1$ is not semi strong. Therefore, $w$ is adjacent with only $u_i$. Therefore, $u_i \in w$ is a component of $V_1$. Further $w$ is not adjacent with any vertex of $V_2$. If $w$ is adjacent with $w_1 \in V_2$, then $r(w_1) = (1, 1)$, a contradiction. If $V_1$ contains a third vertex $x$ distinct from $u_i$ and $w$. Then $x$ is adjacent to some vertex of $V_2$, a contradiction, since that vertex $w_1$ and $v_2$ have the same coordinate. Therefore $|V_1| = 2$. If $|V_2| \geq 2$, then proceeding as before $|V_2| = 2$ and $G = P_2$. But $\chi_{spd}(P_2) = 3$, a contradiction. Therefore $|V_2| = 1$. Hence $|V_2| = 1$ we get $G = P_3$. If $|V_1| = 2$, then $G = P_2$ or $P_3$. Therefore $\chi_{spd}(G) = 2$ if and only if $G = P_2$ or $P_3$.

**Theorem 1.6.** Let $G$ be a graph with full degree vertex, say $u$. Then $\chi_{spd}(G) = n$ if and only if the subgraph induced by a vertex of $G$ other than $u$ has no isolates.

**Proof:** Suppose $G$ has a full degree vertex $u$. Let $v_1, v_2, \ldots, v_{n-1}$ are the vertices of $G$ adjacent with $u$. Then no two vertices $v_i, v_j, i \neq j$ belong to the same color class of a resolving semi strong color partition of $G$.

**Case 1:** Suppose the subgraph induced by vertices of $G$ other than $u$ has an isolate say $v_1$.

Let $\Pi = \{\{u, v_1\}, \{v_2\}, \ldots, \{v_{n-1}\}\}$. By a hypothesis, $v_1$ is adjacent with $u$ for every $i$, $v_i$ is not adjacent with $v_1$ for any $i$. Clearly $u, v_1$ are resolved by any $v_i$, ($i \geq 2$). Therefore $\chi_{spd}(G) = n - 1$.

**Case 2:** Suppose the subgraph induced by vertices of $G$ other than $u$ has no isolate.

Then $\Pi$ is not a semi strong color partition of $G$. Therefore, $\Pi = \{\{u\}, \{v_1\}, \ldots, \{v_{n-1}\}\}$ is a resolving semi strong color partition of $G$ and it is minimum. That is $\chi_{spd}(G) = n$.

**Theorem 1.7.** Let $G$ be a connected graph. $\chi_{spd}(G) = n$ if and only if $N(G) = K_n$.

**Proof:** Let $\chi_{spd}(G) = n$. Let $V(G) = \{u_1, u_2, \ldots, u_k\}$. Suppose $\text{diam}(G) = k \geq 3$.

Let $u = u_1, u_2, \ldots, u_{k+1} = v$ be a diametrical path in $G$. Let $\Pi = \{\{u, v\}, \{V_1\}, \ldots, \{V_{n-1}\}\}\{\{u, v\}, \{V_2\}, \ldots, \{V_{n-1}\}\}$ where $V_2, \ldots, V_{n-1}$ are singletons $u$ and $v$ are resolved by $\{u_2\}$. Then $\Pi$ is a resolving semi strong color partition of $G$. Therefore, $\chi_{spd}(G) \leq n - 1$, a contradiction. Therefore, $\text{diam}(G) \leq 2$. Suppose $u_1$ and $u_2$ are adjacent and $u_1u_2$ is not the edge of a triangle. Let $\Pi = \{\{u_1, u_2\}, V_2, \ldots, V_{n-1}\}$ where $V_2, \ldots, V_{n-1}, u_1$ and $u_2$ are resolved by $\{u_3\}$ where $u_1$ is adjacent with $u_2$ and $u_2$ is not adjacent with $u_3$. Then $\Pi$ is a resolving semi strong color partition of $G$, a contradiction.
Let $|V(G)| \geq 4$. If $u_1$ and $u_2$ are adjacent. Then $u_1u_2$ is an edge of triangle. Therefore, $N(G) = K_n$. Suppose $|V(G)| = 3$. Then $G = P_3$ or $K_3$. $\chi_{spd}(P_3) = 2 < 3$. Therefore, $G = K_3$. Therefore, $N(G) = K_n$. The converse is obvious.

**Theorem 1.8.** For $m \geq 3$, $\chi_{spd}(K_{a_1, a_2, \ldots, a_m}) = a_1 + a_2 + \ldots + a_m$.

**Proof:** $\chi(K_{a_1, a_2, \ldots, a_m}) = a_1 + a_2 + \ldots + a_m$. Further $\chi(K_{a_1, a_2, \ldots, a_m}) \leq \chi_{spd}(K_{a_1, a_2, \ldots, a_m})$. Therefore, $\chi_{spd}(K_{a_1, a_2, \ldots, a_m}) = a_1 + a_2 + \ldots + a_m$.

**Theorem 1.9.** Let $G = K_m(a_1, a_2, \ldots, a_m)$. Then $\chi_{spd}(G) = m + \max\{a_i\}, 1 \leq i \leq m$.

**Proof:** Let $G = K_m(a_1, a_2, \ldots, a_m)$. Let $V(G) = \{u_1, u_2, \ldots, u_m, u_{1,1}, u_{1,2}, \ldots, u_{1,a_1}, \ldots, u_{m,1}, u_{m,2}, \ldots, u_{m,a_m}\}$ where $V(K_m) = \{u_1, u_2, \ldots, u_m\}$. Let $\Pi = \{\{u_1\}, \{u_2\}, \ldots, \{u_m\}, \{u_{1,1}, u_{2,1}, \ldots, u_{m,1}\}, \ldots\}$. Then $\Pi$ is a resolving semi strong color partition of $G$. Therefore, $\chi_{spd}(G) \leq m + \max\{a_i\}$, $1 \leq i \leq m$. Since $\chi(G) \leq \chi_{spd}(G)$ and $\chi(G) = m + \max\{a_i\}$, $1 \leq i \leq m$. Therefore, $\chi_{spd}(G) = m + \max\{a_i\}$, $1 \leq i \leq m$.

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