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COMPATIBLE MAPPING AND COMMON FIXED POINT FOR FIVE MAPPINGS

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ABSTRACT

In this paper, it is proved that the existence of unique common fixed point theorem involving for five mappings with semi-compatibility, weak compatibility and commutativity on Metric space. This result improves and generalizes some known result of Imdad and Khan [7] by using functional expressions.

Subject Classification. Primary 54H25 , Secondary 47H10

Keywords*- Fixed point, Complete metric space, semi-compatibility and weak compatibility mappings.*

I. INTRODUCTION

The study of common fixed point of mapping satisfying different contraction condition has been a very active field of research activity and may be extended to the abstract spaces. Fisher[4,5] generalizes affixed point theorem of Jungck[6]. Hicks and Kubicek [1] proved the Mann iteration process in Hilbert space. Pandhare and Waghmode [9] proved a common fixed point theorem in Hilbert space. Srinivas .V [11] proved a common fixed point theorem on compatible mappings of type (p) . Shrivastava [12] a proved compatible mapping and common fixed point theorem. Gupta [13] Common fixed point theorem for compatible mappings of type (A-1) in complete fuzzy metric space. Sessa [10] introduced the notion of weak commutativity which asserts that a pair of self mapping (A,B) on a metric space (X, d) is said to be weakly commuting if d(ABx, BAx) $\leq$ d(Bx, Ax) for all x in X. Motivated by Sessa [10], The notion of compatible mapping was introduced by Jungck [7] , which asserts that a pair self mapping (A,B) of a metric space (X, d) is said to be compatible if $\lim_{n\to\infty}$ (ABx_n, BAx_n) = 0 whenver $\lim_{n\to\infty}$ Ax_n= $\lim_{n\to\infty}$ Bx_n = t □□ X. A weakly commuting pair is compatible, but not conversely as demonstrated in Jungck [7]. Lohani and Badshah [8] proved some common fixed point theorem for four compatible mappings on Metric space ,Imdad [2] proved a unique common fixed point theorem on five mappings.

Definition 1. Let *S* and *T* be mappings from a metric space *(X,d)* into itself. Then mappings *S* and *T* are said to be *compatible* if $\lim_{n\to\infty} d(STx_n, TSx_n) = 0$ whenever $\{x_n\}$ is a sequence in *X* such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

Definition 2. Let *S* and *T* be mappings from a metric space *(X,d)* into itself. Then mappings *S* and *T* are said to be *weakly compatible* if they commute at their coincidence point that is *STx=TSx* whenever *Sx=Tx*, $x \in X$.

Definition 3. Let *S* and *T* be mappings from a metric space *(X,d)* into itself. Then mappings *S* and *T* are said to be *semi-compatible* if $\lim_{n\to\infty} d(STx_n, Tx_n) = 0$ whenever $\{x_n\}$ is a sequence in *X* such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

Note that compatible mappings are weakly compatible but weakly compatible mappings are not necessarily compatible and clearly the pair (S,T) is semi-compatible then they are weakly compatible.

In this paper we prove a common fixed point theorem involving five mappings which generalizes earlier result due to Imdad and Khan [3] by improving contraction condition besides optimally chosen suitable semi compatible, weak compatible and commuting condition on Complete Metric space by using a rational inequality.

Theorem 1 . Let *A, B, S, T* and *P* be self mappings of complete metric space *(X,d)* satisfying the $AB(X) \subset P(X)$, $ST(X) \subset P(X)$ and $AB(X) \cap ST(X) \subset P(X)$ and

$$d(ABx, STy) \le \alpha_1 \left[\frac{d(ABx, Px)\{1 + d(STy, Py)\}}{\{1 + d(Px, Py)\}} \right] + \alpha_2 [d(ABx, Py) + d(STy, Px)] + \alpha_3 d(Px, Py) \quad \textbf{(1)}$$

for each $x, y \in X$ and $\alpha_1, \alpha_2, \alpha_3 \ge 0$, $\alpha_1 + 2\alpha_2 + \alpha_3 < 1$ either if ,

(a){AB, P} are semi-compatible, P or AB is continuous and (ST,P) are weakly compatible or
(b){ST, P} are semi-compatible P or ST is continuous and (AB, P) are weakly compatible. Then AB, ST and P have a unique common fixed point. Furthermore if the pairs (A,B),(A,P),(B,P),(S,T),(S,P)and (T,P) are commuting mapping then A,B,S,T and P have a unique common fixed point.

Proof. Let x_0 be an arbitrary point in X, since $AB(X) \subset P(X)$ we can find a point x_1 in X such that $ABx_0 = Px_1$. Also since $ST(X) \subset P(X)$ we can choose a point x_2 with $STx_1 = Ix_2$, using this argument repeatedly one can construct a sequence $\{z_n\}$ such that
$z_{2n} = ABx_{2n} = Px_{2n+1}$, $z_{2n+1} = STx_{2n+1} = Px_{2n+2}$ for n = 0, 1, 2, … .
$d(z_{2n+2}, z_{2n+1}) = d(ABx_{2n+2}, STx_{2n+1})$

$$\le \alpha_1 \left[\frac{d(ABx_{2n+2}, Px_{2n+2})\{1 + d(STx_{2n+1}, Px_{2n+1})\}}{\{1 + d(Px_{2n+2}, Px_{2n+1})\}} \right]$$

$$+ \alpha_2 [d(ABx_{2n+2}, Px_{2n+1}) + d(STx_{2n+1}, Px_{2n+2})] + \alpha_3 d(Px_{2n+2}, Px_{2n+1})$$

$$\le \alpha_1 \left[\frac{d(z_{2n+2}, z_{2n+1})\{1 + d(z_{2n+1}, z_{2n})\}}{\{1 + d(z_{2n+1}, z_{2n})\}} \right]$$

$$+ \alpha_2 [d(z_{2n+2}, z_{2n}) + d(z_{2n+1}, z_{2n+1})] + \alpha_3 d(z_{2n+1}, z_{2n})$$

$$\le \alpha_1 [d(z_{2n+2}, z_{2n+1})] + \alpha_2 [d(z_{2n+2}, z_{2n})] + \alpha_3 d(z_{2n+1}, z_{2n})$$

$$d(z_{2n+2}, z_{2n+1}) \le \frac{\alpha_2 + \alpha_3}{(1 - \alpha_1 - \alpha_2)} d(z_{2n+1}, z_{2n}) \qquad \text{where } k = \frac{\alpha_2 + \alpha_3}{(1 - \alpha_1 - \alpha_2)} < 1.$$

Thus for every n we have,

$$d(z_{n+1}, z_n) \le k\, d(z_n, z_{n-1}) \qquad \text{where} \quad k = \frac{\alpha_2 + \alpha_3}{(1 - \alpha_1 - \alpha_2)} < 1 \quad \textbf{(2)}$$

which shows that $\{z_n\}$ is a Cauchy sequence in the Metric space *(X,d)* and so has a limit point z in *X*. Hence the sequence $ABx_{2n} = Px_{2n+1}$ and $STx_{2n+1} = Px_{2n+2}$ which are subsequences also converge to the point z.

Let us now assume that P is continuous so that the sequences $\{P^2x_{2n}\}$ and $\{PABx_{2n}\}$converges to Pz and also in view of semi-compatibility of {AB,P}, $\{ABPx_{2n}\}$ converges to Pz.
Now put $x = Px_{2n}$ and $y = x_{2n+1}$ in equation (1), we have

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$$d(ABPx_{2n},STx_{2n+1})\le\alpha_1\left[\frac{d(ABPx_{2n},P^2x_{2n})\{1+d(STx_{2n+1},Px_{2n+1})\}}{\{1+d(P^2x_{2n},Px_{2n+1})\}}\right]$$

$$+\alpha_2[d(ABPx_{2n},Px_{2n+1})+d(STx_{2n+1}P^2x_{2n},)]+\alpha_3 d(P^2x_{2n},Px_{2n+1})$$

letting n→∞ we have

$$d(Pz,z)\le\alpha_1\left[\frac{d(Pz,Pz)\{1+d(z,z)\}}{\{1+d(Pz,z)\}}\right]+\alpha_2[d(Pz,z)+d(z,Pz)]+\alpha_3 d(Pz,z)$$

$$d(Pz,z)\le(2\alpha_2+\alpha_3)d(Pz,z)$$

so that $Pz=z$

Now put x=z and y=x_{2n+1} in equation (1)

$$d(ABz,STx_{2n+1})\le\alpha_1\left[\frac{d(ABz,Pz)\{1+d(STx_{2n+1},Px_{2n+1})\}}{\{1+d(Pz,Px_{2n+1})\}}\right]$$

$$+\alpha_2[d(ABz,Px_{2n+1})+d(STx_{2n+1},Pz)]+\alpha_3 d(Pz,Px_{2n+1})$$

letting n→∞ we have

$$d(ABz,z)\le\alpha_1\left[\frac{d(ABz,z)\{1+d(z,z)\}}{\{1+d(z,z)\}}\right]+\alpha_2[d(ABz,z)+d(z,z)]+\alpha_3 d(z,z)$$

$$d(ABz,z)\le(\alpha_1+\alpha_2)d(ABz,z)$$

so that $ABz=z$.

Since AB(X) ⊂ P(X) there always exists a point z' such that Pz' = z so that STz = ST(Pz').

Now put x = x_{2n} and y = z' in equation (1),

$$d(ABx_{2n},STz')\le\alpha_1\left[\frac{d(ABx_{2n},Px_{2n})\{1+d(STz',Pz')\}}{\{1+d(Px_{2n},Pz')\}}\right]$$

$$+\alpha_2[d(ABx_{2n},Pz')+d(STz',Px_{2n})]+\alpha_3 d(Px_{2n},Pz')$$

letting n→∞ we have

$$d(z,STz')\le\alpha_1\left[\frac{d(z,z)\{1+d(STz',z)\}}{\{1+d(z,z)\}}\right]+\alpha_2[d(z,z)+d(STz',z)]+\alpha_3 d(z,z)$$

$$(1-\alpha_2)d(STz',z)\le 0$$

so that $STz'=z$.

Hence STz' = z = Pz 'which shows that z' is the coincidence point of ST and P.

Now using the weak compatibility of (ST, P), we have

STz = ST (Pz') = P(STz') = Pz, which shows that z is also a coincidence point of the pair (ST,P).

Now put x = z and y = z in equation (1)

$$d(ABz,STz)\le\alpha_1\left[\frac{d(ABz,Pz)\{1+d(STz,Pz)\}}{\{1+d(Pz,Pz)\}}\right]+\alpha_2[d(ABz,Pz)+d(STz,Pz)]+\alpha_3 d(Pz,Pz)$$

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$$d(z,STz)\le \alpha_1\left[\frac{d(z,z)\{1+d(STz,z)\}}{\{1+d(z,z)\}}\right]+\alpha_2[d(z,z)+d(STz,z)]+\alpha_3 d(z,z)$$

$$(1-\alpha_2)d(STz,z)\le 0$$

so that $STz = z$. Hence z = STz = Pz which shows that z is common fixed point of AB, ST and P.

Now suppose that AB is continuous so that the sequence $\{AB^2x_{2n}\}$ and $\{ABPx_{2n}\}$ converges ABz .Since (AB,P) is semi-compatible it follows that $\{PABx_{2n}\}$ also converges to ABz.

Thus put $x = ABx_{2n}$ and $y = x_{2n+1}$ in equation (1) we have

$$d(AB^2x_{2n},STx_{2n+1})\le \alpha_1\left[\frac{d(AB^2x_{2n},PABx_{2n})\{1+d(STx_{2n+1},Px_{2n+1})\}}{\{1+d(PABx_{2n},Px_{2n+1})\}}\right]$$

$$+\alpha_2[d(AB^2x_{2n},Px_{2n+1})+d(STx_{2n+1},PABx_{2n})]+\alpha_3 d(PABx_{2n},Px_{2n+1})$$

letting n→∞ we have

$$d(ABz,z)\le \alpha_1\left[\frac{d(ABz,ABz)\{1+d(z,z)\}}{\{1+d(ABz,z)\}}\right]+\alpha_2[d(ABz,z)+d(z,ABz)]+\alpha_3 d(ABz,z)$$

$$(1-2\alpha_2-\alpha_3)d(ABz,z)\le 0$$

so that $ABz = z$.

Let there exist z' in X such that ABz = z= Pz'.
Then put x = ABx_{2n} and y = z' in equation (1)

$$d(AB^2x_{2n},STz')\le \alpha_1\left[\frac{d(AB^2x_{2n},PABx_{2n})\{1+d(STz',Pz')\}}{\{1+d(PABx_{2n},Pz')\}}\right]$$

$$+\alpha_2[d(AB^2x_{2n},Pz')+d(STz',PABx_{2n})]+\alpha_3 d(PABx_{2n},Pz')$$

letting n→∞ we have

$$d(ABz,STz')\le \alpha_1\left[\frac{d(ABz,ABz)\{1+d(STz',z)\}}{\{1+d(ABz,z)\}}\right]+\alpha_2[d(ABz,z)+d(STz',ABz)]+\alpha_3 d(ABz,z)$$

$$(1-\alpha_2)d(z,STz')\le 0$$

so that $STz' = z$.

This gives STz' = z = Pz' Thus z' is a coincidence point of (ST,P) since the pair (ST,P) is weakly compatible one has STz =ST (Pz') = Pz which show that STz = Pz.
Put $x = x_{2n}$ and y = z in equation (1) we have

$$d(ABx_{2n},STz)\le \alpha_1\left[\frac{d(ABx_{2n},Px_{2n})\{1+d(STz,Pz)\}}{\{1+d(Px_{2n},Pz)\}}\right]$$

$$+\alpha_2[d(ABx_{2n},Pz)+d(STz,Px_{2n})]+\alpha_3 d(Px_{2n},Pz)$$

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letting n→∞ we have

$$d(z,STz)\le \alpha_1\left[\frac{d(z,z)\{1+d(STz,z)\}}{\{1+d(z,z)\}}\right]+\alpha_2[d(z,z)+d(STz,z)]+\alpha_3 d(z,z)$$

$$(1-\alpha_2)d(z,STz)\le 0$$

which implies $STz=z$

so that STz = z =Pz.

The point z therefore is in range of ST and since ST(X) ⊂ P(X) there exists a point z'' in X such that Pz'' = z. Thus put x = z'' and y = z in equation (1)

$$d(ABz'',STz)\le \alpha_1\left[\frac{d(ABz'',Pz'')\{1+d(STz,Pz)\}}{\{1+d(Pz'',Pz)\}}\right]$$

$$+\alpha_2[d(ABz'',Pz)+d(STz,Pz'')]+\alpha_3 d(Pz'',Pz)$$

$$d(ABz'',z)\le \alpha_1\left[\frac{d(ABz'',z)\{1+d(z,z)\}}{\{1+d(z,z)\}}\right]\alpha_2[d(ABz'',z)+d(z,z)]+\alpha_3 d(z,z)$$

$$(1-\alpha_2)d(ABz'',z)\le 0$$

which implies $ABz''=z$

Also since (AB,P) are semi-compatible are hence weakly commuting we obtain ABz = Pz = z Thus we have proved that z is a common fixed point of AB, ST and P.

If mappings ST or P is continuous instead of AB or P, then the proof that z is a common fixed point of AB,ST and P is similar.

Let v be another fixed point of P, AB and ST then v =Pv =ABv =STv

$$d(ABz,STv)\le \alpha_1\left[\frac{d(ABz,Pz)\{1+d(STv,Pv)\}}{\{1+d(Pz,Pv)\}}\right]+\alpha_2[d(ABz,Pv)+d(STv,Pz)]+\alpha_3 d(Pz,Pv)$$

$$d(z,v)\le \alpha_1\left[\frac{d(z,z)\{1+d(v,v)\}}{\{1+d(z,v)\}}\right]+\alpha_2[d(z,v)+d(v,z)]+\alpha_3 d(z,v)$$

$$d(z,v)\le (2\alpha_2+\alpha_3)d(z,v)$$

which implies z = v.

Finally we now show that z is also a common fixed point of the family F={A,B,S,T,P}. When the pairs (A,B),(A,P),(B,P),(S,T),(S,P)and (T,P) are commuting pairs. For this event we write,

Az=A(ABz) = A(BA)z = AB(Az)

Az = A(Pz) =AP(z)=PA(z) =P(Az)

Bz = B(ABz) = BA (Bz) = AB (Bz)

Bz = B(Pz)= BP(z)= PB(z)= P (Bz)

Sz=S(STz) = S(TS)z = ST(Sz)

Sz = S(Pz) =SP(z)=PS(z) =P(Sz)

Tz = T(STz) = TS (Tz) = ST (Tz)

Tz = T(Pz)= TP(z)= PT(z)= P (Tz)

which shows that Az and Bz are common fixed point of (AB,P), yielding thereby Az =Bz =Pz = ABz . where as Sz and Tz are common fixed point of (ST,P) it also shows that Sz = z = Tz = Pz =STz.
Now we need to show that Az = Sz (Bz = Tz) also remains a common fixed point of both the pairs (AB,P) and (ST,P). For this

d(Az, Sz) = d(A(BAz),S (TSz)) = d(AB(Az) ,ST (Sz))

$$\le \alpha_1\left[\frac{d(AB(Az),P(Az))\{1+d(ST(Sz),P(Sz))\}}{\{1+d(P(Az),P(Sz))\}}\right]$$

$$+\alpha_2[d(AB(Az),P(Sz))+d(ST(Sz),P(Az))]+\alpha_3 d(P(Az),P(Sz))$$

Implies that (1-2□$_2$)d(Az,Sz) $\le$ 0 so that Az = Sz.
Similarly it can be show that Bz=Tz, Thus z is the unique common fixed point of A,B, S ,T and P.

Example. Let A, B, S,T and P be self mapping of Hilbert space H. Let X= [0,1] be a closed subset of H. We define mapping

$Ax=\frac{3}{4}x$, $Bx=\frac{4}{9}x$, $Sx=\frac{2}{3}x$, $Tx=\frac{3}{10}x$ and $Px=\frac{1}{3}x$.

Clearly $AB(X)=\left[0,\frac{1}{3}\right]\subset P(X)=\left[0,\frac{1}{3}\right]$ and $ST(X)=\left[0,\frac{1}{5}\right]\subset P(X)=\left[0,\frac{1}{3}\right]$ and

$$AB(X)\cap ST(X)=\left[0,\frac{1}{3}\right]\cap\left[0,\frac{1}{5}\right]\subset P(X)=\left[0,\frac{1}{3}\right]$$

so that $AB(X)\cap ST(X)=\left[0,\frac{1}{5}\right]\subset P(X)=\left[0,\frac{1}{3}\right]$.

Also the pair (AB, P) (ST, P), (A,B), (S,T), (A,P), (B,P), (S,P) and (T,P) are commuting and semi-compatible or weak compatible.

For all x,y in X (x>y) with $\alpha_1=\frac{1}{9}$ and $\alpha_2=\frac{1}{2}$ we have ,

$$\left|\frac{1}{3}x-\frac{1}{5}y\right|\le\alpha_1\left[\frac{\left|\frac{1}{3}x-\frac{1}{3}x\right|\left\{1+\left|\frac{1}{5}y-\frac{1}{3}y\right|\right\}}{\left\{1+\left|\frac{1}{3}x-\frac{1}{3}y\right|\right\}}\right]+\alpha_2\left[\left|\frac{1}{3}x-\frac{1}{3}y\right|+\left|\frac{1}{5}y-\frac{1}{3}x\right|\right]+\alpha_3\left|\frac{1}{3}x-\frac{1}{3}y\right|.$$

Using $\frac{1}{5}y<\frac{1}{3}y$ we get ,

$$\left|\frac{1}{3}x-\frac{1}{5}y\right|\le(2\alpha_2+\alpha_3)\left|\frac{1}{3}x-\frac{1}{5}y\right|$$

which verifies the contraction condition (1).

Clearly 0 is unique common fixed point of A, B, S, T and P.

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